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A SPECTRAL THEORY FOR DISCONTINUOUS STURM-LIOUVILLE PROBLEMS ON THE WHOLE LINE

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In this study, we consider the singular discontinuous Sturm-Liouville problem on the whole line with transmission conditions. For this problem the existence of a spectral matrix-valued function is proved. A Parseval equality and an expansion formula are given for such a problem.

1. Introduction

The eigenfunction expanding theorems play an important role in solving boundary value problems related to partial differential equations When the method of separation of variables is applied to partial differential equations, we get a Sturm-Liouville problem. Thus, we consider the problem of expanding an arbitrary function as a series of eigenfunctions. Such problems for various differential operators were investigated by a number of authors (see [5]-[9], [12]-[13], [16], [24] and the references therein).

The discontinuous Sturm-Liouville problems have been discussed for a long time and several results on these problems have been obtained (see, e.g., [1-5, 7, 10-13, 15, 17-23, 25-28] and the references therein). Such problems arise

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in the theory of heat and mass transfer (see [18]), radio science (see [19]) and geophysics (see [15]). In the literature, there is much research on regular discontinuous Sturm-Liouville problems (see [1, 11, 20-23, 26-28]). However, the related results for the singular problem are not so rich (see [2-4, 10, 25]). In [17], Li, Sun and Hao investigated a singular Sturm-Liouville problem with transmission conditions at finite interior points. They gave a definition of Weyl function for such problems in the case of the limit circle case. In [5], the authors studied a singular Sturm-Liouville problem with transmission condition on semi-infinite interval. They established an expansion formula in eigenfunctions in terms of the spectral function.

In this paper, we study a singular discontinuous Sturm-Liouville problem on the whole line with transmission conditions. We prove the existence of a spectral matrix-valued function for such a problem. A Parseval equality and an expansion formula are given in this problem on the whole line.

2. Main Results

Let us consider the Sturm-Liouville expression

$$\tau(y) := -(p(x)y')' + q(x)y,\ x \in (a,c) \cup (c,b),$$

where $I_1 := [a,c)$, $I_2 := (c,b]$, $-\infty < a < 0 < c < b < \infty$ and $I := I_1 \cup I_2$. We assume that the points a, b and c are regular for the differential expression τ. On the other hand, p and q are real-valued, Lebesgue measurable functions on I and $\frac{1}{p}, q \in L^1(I_k)$, $k = 1,2$. The point c is regular if $\frac{1}{p}, q \in L^1[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon > 0$. Now we consider the Sturm-Liouville equation

$$\tau(y) = \lambda y,\ \ x \in I, \tag{1}$$

with the boundary conditions

$$y(a)\cos\beta + \left(py'\right)(a)\sin\beta = 0,\ \beta \in \mathbb{R} := (-\infty,\infty), \tag{2}$$

$$y(b)\cos\alpha + \left(py'\right)(b)\sin\alpha = 0,\ \alpha \in \mathbb{R}, \tag{3}$$

and transmission conditions

$$Y(c+) = CY(c-),\ Y = \begin{pmatrix} y \\ y' \end{pmatrix}, C \in M_2(\mathbb{R}),\ \det C = \delta > 0, \tag{4}$$

where $M_2(\mathbb{R})$ denotes the set of all 2×2 matrices with entries from $\mathbb{R}$.

Now, we introduce the Hilbert space $H_1 = L^2(I_1) \dot{+} L^2(I_2)$ with the inner product

$$\langle f,g\rangle_{H_1} := \int_a^c f^{(1)}\overline{g^{(1)}}dx + \gamma\int_c^b f^{(2)}\overline{g^{(2)}}dx,\ \gamma = \frac{1}{\delta},$$

where

$$f(x)=\begin{cases} f^{(1)}(x), & x\in I_1 \\ f^{(2)}(x), & x\in I_2 \end{cases},\quad g(x)=\begin{cases} g^{(1)}(x), & x\in I_1 \\ g^{(2)}(x), & x\in I_2. \end{cases}$$

Let us denote by D the linear set of all functions $y\in H_1$ such that y and py' are locally absolutely continuous functions on I, the one-sided limits $y(c\pm),(py')(c\pm)$ exist and are finite, and $\tau(y)\in H_1$. The operator T defined by $Ty=\tau(y)$ is called the *maximal operator* T on H_1.

For two arbitrary functions $y,z\in D$, we have the Green's formula given by

$$\int_a^b \tau(y)\,\bar{z}dx-\int_a^b y\overline{\tau(z)}dx=[y,z]_{c-}-[y,z]_a+[y,z]_b-[y,z]_{c+}, \tag{5}$$

where $[y,z]_x=y(x)\overline{(pz')(x)}-(py')(x)\overline{z(x)}\ \ (x\in I)$.

We will denote by

$$\phi_i(x,\lambda)=\begin{cases} \phi_i^{(1)}(x,\lambda), & x\in I_1 \\ \phi_i^{(2)}(x,\lambda), & x\in I_2 \end{cases},\ i=1,2$$

the solutions of the equation defined by (1) which satisfy the initial conditions

$$\phi_1(0,\lambda)=0,\ \left(p\phi_1'\right)(0,\lambda)=1,\ \phi_2(0,\lambda)=1,\ \left(p\phi_2'\right)(0,\lambda)=0, \tag{6}$$

and transmission conditions

$$\Phi_i(c+,\lambda)=C\Phi_i(c-,\lambda),\ \Phi_i=\begin{pmatrix} \phi_i(x,\lambda) \\ (p\phi_i')(x,\lambda) \end{pmatrix}, \tag{7}$$

$$C\in M_2(\mathbb{R}),\ \det C=\delta>0,\ i=1,2.$$

In [11-13, 26, 27] the authors proved that the regular boundary value problem defined by (1)-(4) is self-adjoint and has a compact resolvent, so it has a purely discrete spectrum.

Let $\lambda_n\ (n=1,2,...)$ be the (real) eigenvalues and

$$y_n(x)=\begin{cases} y_n^{(1)}(x), & x\in I_1 \\ y_n^{(2)}(x), & x\in I_2 \end{cases}\quad (n=1,2,...)$$

the corresponding real-valued eigenfunctions of the problem defined by (1)-(4). Since the solutions ϕ_1 and ϕ_2 are linearly independent, we get

$$y_n(x)=c_n\phi_1(x,\lambda_n)+d_n\phi_2(x,\lambda_n)\ \ (n=1,2,...).$$

Let $f \in H_1$ be a real-valued function. If we apply the Parseval equality (see [12], [13]) to the function f, then we obtain

$$\int_a^c \left(f^{(1)}(x)\right)^2 dx + \gamma \int_c^b \left(f^{(2)}(x)\right)^2 dx$$

$$= \sum_{n=1}^{\infty} \left\{ \int_a^c f^{(1)}(x) y_n^{(1)}(x)\, dx + \gamma \int_c^b f^{(2)}(x) y_n^{(2)}(x)\, dx \right\}^2$$

$$= \sum_{n=1}^{\infty} \{\langle f, y_n \rangle_{H_1}\}^2 = \sum_{n=1}^{\infty} \{\langle f(.), c_n \phi_1(., \lambda_n) + d_n \phi_2(., \lambda_n) \rangle_{H_1}\}^2$$

$$= \sum_{n=1}^{\infty} c_n^2 \{\langle f(.), \phi_1(., \lambda_n) \rangle_{H_1}\}^2$$

$$+ 2 \sum_{n=1}^{\infty} c_n d_n \prod_{j=1}^{2} \{\langle f(.), \phi_j(., \lambda_n) \rangle_{H_1}\}$$

$$+ \sum_{n=1}^{\infty} d_n^2 \{\langle f(.), \phi_2(., \lambda_n) \rangle_{H_1}\}^2. \tag{8}$$

Now, we will define the step function $\mu_{ij,a,b}$ $(i, j = 1, 2)$ on (a, b) by

$$\mu_{11,a,b}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} c_n^2, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} c_n^2, & \text{for } \lambda > 0, \end{cases}$$

$$\mu_{12,a,b}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} c_n d_n, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} c_n d_n, & \text{for } \lambda > 0, \end{cases},$$

$$\mu_{12,a,b}(\lambda) = \mu_{21,a,b}(\lambda),$$

$$\mu_{22,a,b}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} d_n^2, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} d_n^2, & \text{for } \lambda > 0. \end{cases}$$

From the equality (8), we obtain

$$\int_a^c \left(f^{(1)}(x)\right)^2 dx + \gamma \int_c^b \left(f^{(2)}(x)\right)^2 dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda)\, d\mu_{ij,a,b}(\lambda), \tag{9}$$

where

$$F_i(\lambda) = \langle f(.), \phi_i(., \lambda) \rangle_{H_1} \ \ (i = 1, 2).$$

A function f defined on an interval $[a_1, b_1]$ is said to be of *bounded variation* if there is a constant $C > 0$ such that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq C$$

for every partition

$$a_1 = x_0 < x_1 < \ldots < x_n = b_1$$

of $[a_1, b_1]$ by points of subdivision $x_0, x_1, \ldots, x_n$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[a_1, b_1]$, denoted by $\bigvee_{a_1}^{b_1}(f)$, we mean the quantity

$$\bigvee_{a_1}^{b_1}(f) := \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions of the interval $[a_1, b_1]$ (see [14]).

Now, we will prove a lemma.

Lemma 2.1. *There exists a positive constant* $\Lambda = \Lambda(\xi)$, $\xi > 0$ *such that*

$$\bigvee_{-\xi}^{\xi}\{\mu_{ij,a,b}(\lambda)\} < \Lambda \ (i, j = 1, 2), \tag{10}$$

where Λ *does not depend on a and b.*

Proof. Since the functions $\phi_i^{[j-1]}(x, \lambda)$ $(i, j = 1, 2)$ (where $y^{[1]}$ denotes py' and $y^{[0]}$ denotes y) are both continuous with respect to $x \in [a, c)$ and $\lambda \in \mathbb{R}$, it follows from (6) that

$$\phi_i^{[j-1]}(0, \lambda) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Thus for any $\varepsilon > 0$ there exists a k with $0 < k < c$ such that

$$\left|\phi_i^{[j-1]}(x, \lambda) - \delta_{ij}\right| < \varepsilon, \ \varepsilon > 0, \ |\lambda| < \xi, \ x \in [0, k]. \tag{11}$$

Let $f_k(.)$ be a nonnegative function such that $f_k(.)$ vanishes outside the interval $(0, k)$ with the property

$$\int_0^k f_k(x)\, dx = 1, \tag{12}$$

and let $f_k^{[1]}(x)$ be a continuos function on $[a, c)$.

Now, if we apply the Parseval equality (9) to the functions $f_k^{[s-1]}(x)\ (s=1,2)$, then we get

$$\int_0^k (f_k^{[s-1]}(x))^2 dx \geq \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_{is}(\lambda) F_{js}(\lambda)\, d\mu_{ij,a,b}(\lambda),$$

where

$$F_{i1}(\lambda) = \int_0^k f_k(x)\, \phi_i(x,\lambda)\, dx,$$

and

$$F_{i2}(\lambda) = \int_0^k f_k^{[1]}(x)\, \phi_i(x,\lambda)\, dx = -\int_0^k f_k(x)\, \phi_i^{[1]}(x,\lambda)\, dx.$$

By using the inequality (11) and the equality (12), we obtain

$$|F_{is}(\lambda) - \delta_{is}| < \varepsilon,\ i,s = 1,2,\ |\lambda| < \xi. \tag{13}$$

Now by applying the Parseval equality (9) to the functions $f_k^{[s-1]}(x)\ (s=1,2)$, we get

$$\int_0^k (f_k^{[s-1]}(x))^2 dx \geq \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} (\delta_{is} - \varepsilon)(\delta_{js} - \varepsilon)\, d\mu_{ij,a,b}(\lambda). \tag{14}$$

If we take $s=1$ in the inequality (14), we have

$$\int_0^k f_k^2(x)\, dx \geq (1-\varepsilon)^2 \int_{-\xi}^{\xi} d\mu_{11,a,b}(\lambda) + \varepsilon(1+\varepsilon) \int_{-\xi}^{\xi} d\mu_{12,a,b}(\lambda)$$

$$+\varepsilon(1+\varepsilon) \int_{-\xi}^{\xi} d\mu_{21,a,b}(\lambda) + \varepsilon^2 \int_{-\xi}^{\xi} d\mu_{22,a,b}(\lambda)$$

$$= (1-\varepsilon)^2 (\mu_{11,a,b}(\xi) - \mu_{11,a,b}(-\xi))$$

$$+2\varepsilon(1+\varepsilon) \bigvee_{-\xi}^{\xi} \{\mu_{12,a,b}(\lambda)\} + \varepsilon^2 (\mu_{22,a,b}(\xi) - \mu_{22,a,b}(-\xi)).$$

Since

$$\bigvee_{-\xi}^{\xi} \{\mu_{12,a,b}(\lambda)\} \leq \frac{1}{2} [\mu_{11,a,b}(\xi) - \mu_{11,a,b}(-\xi) + \mu_{22,a,b}(\xi) - \mu_{22,a,b}(-\xi)], \tag{15}$$

we get

$$\int_0^k f_k^2(x)\,dx \geq \left(2\varepsilon^2 - 3\varepsilon + 1\right)\{\mu_{11,a,b}(\xi) - \mu_{11,a,b}(-\xi)\} \\ + 2\varepsilon(\varepsilon - 1)\{\mu_{22,a,b}(\xi) - \mu_{22,a,b}(-\xi)\}. \tag{16}$$

Putting $s = 2$ in (14), we get

$$\int_0^k (f_k^{[1]}(x))^2 dx \geq \left(2\varepsilon^2 - 3\varepsilon + 1\right)\{\mu_{22,a,b}(\xi) - \mu_{22,a,b}(-\xi)\} \\ + 2\varepsilon(\varepsilon - 1)\{\mu_{11,a,b}(\xi) - \mu_{11,a,b}(-\xi)\}. \tag{17}$$

If we add the inequalities (16) and (17), then we get

$$\int_0^k f_k^2(x)\,dx + \int_0^k (f_k^{[1]}(x))^2 dx \\ \geq (2\varepsilon - 1)^2 \left\{ \begin{array}{c} \mu_{11,a,b}(\xi) - \mu_{11,a,b}(-\xi) \\ +\mu_{22,a,b}(\xi) - \mu_{22,a,b}(-\xi) \end{array} \right\}.$$

Hence we obtain the assertion of the lemma for the functions $\mu_{11,a,b}(\lambda)$ and $\mu_{22,a,b}(\lambda)$ relying on their monotonicity. From (15), we get the assertion of the lemma for the function $\mu_{12,a,b}(\lambda)$. □

Now, we recall the following theorems of Helly.

Theorem 2.2 ([14]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing functions on a finite interval $a_1 \leq \lambda \leq b_1$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),\ a_1 \leq \lambda \leq b_1.$$

Theorem 2.3 ([14]). *Assume that $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded sequence of nondecreasing functions on a finite interval $a_1 \leq \lambda \leq b_1$, and suppose that*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),\ a_1 \leq \lambda \leq b_1.$$

If f is any continuous function on $a_1 \leq \lambda \leq b_1$, then

$$\lim_{n\to\infty} \int_{a_1}^{b_1} f(\lambda)\,dw_n(\lambda) = \int_{a_1}^{b_1} f(\lambda)\,dw(\lambda).$$

Let ρ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L_\rho^2(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesgue-Stieltjes measure defined by ρ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\,d\rho(\lambda) < \infty,$$

with the inner product

$$(f,g)_{\rho} := \int_{-\infty}^{\infty} f(\lambda) g(\lambda) d\rho(\lambda).$$

We introduce the Hilbert space $H := L^2(\Omega_1) \dot{+} L^2(\Omega_2)$, $(\Omega_1 := (-\infty, c)$, $\Omega_2 := (c, \infty))$ with the inner product

$$\langle f,g \rangle_H := \int_{-\infty}^{c} f^{(1)} \overline{g^{(1)}} dx + \gamma \int_{c}^{\infty} f^{(2)} \overline{g^{(2)}} dx,$$

where

$$f(x) = \begin{cases} f^{(1)}(x), & x \in \Omega_1 \\ f^{(2)}(x), & x \in \Omega_2 \end{cases}, \quad g(x) = \begin{cases} g^{(1)}(x), & x \in \Omega_1 \\ g^{(2)}(x), & x \in \Omega_2. \end{cases}.$$

The main results of this paper are the following three theorems for the singular Sturm-Liouville equation $\tau(y) = \lambda y$, $(x \in \Omega)$ with the transmission conditions defined by (4).

Theorem 2.4. *Let $f \in H$ be a real-valued function. Then, there exist monotonic functions $\mu_{11}(\lambda)$ and $\mu_{22}(\lambda)$ which are bounded over every finite interval, and a function $\mu_{12}(\lambda)$ which is of bounded variation over every finite interval with the property (Parseval equality)*

$$\int_{-\infty}^{c} \left(f^{(1)}(x)\right)^2 dx + \gamma \int_{c}^{\infty} \left(f^{(2)}(x)\right)^2 dx = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) d\mu_{ij}(\lambda), \tag{18}$$

where

$$F_i(\lambda) = \lim_{n \to \infty} \int_{-n}^{c} f^{(1)}(x) \phi_i^{(1)}(x,\lambda) dx + \gamma \int_{c}^{n} f^{(2)}(x) \phi_i^{(2)}(x,\lambda) dx,$$

$(i = 1,2)$ (generalized Fourier transforms of f).

We note that the matrix-valued function $\mu = (\mu_{ij})_{i,j=1}^{2}$ $(\mu_{12} = \mu_{21})$ is called a *spectral function* for the equation $\tau(y) = \lambda y$, $(x \in \Omega)$ with the transmission conditions defined by (4).

Proof. Assume that the real-valued function

$$f_n(x) = \begin{cases} f_n^{(1)}(x), & x \in \Omega_1 \\ f_n^{(2)}(x), & x \in \Omega_2 \end{cases}$$

satisfies the following conditions:

1) $f_n(x)$ vanishes outside the interval $[-n,c)\cup(c,n]$, where $a<-n<c<n<b$.

2) The functions $f_n(x)$ and $(pf_n')(x)$ have continuous derivatives.

3) $f_n(x)$ satisfies the condition defined by (4).

If we apply the Parseval equality to $f_n(x)$, we get

$$\int_{-n}^{c}(f_n^{(1)}(x))^2dx+\gamma\int_{c}^{n}(f_n^{(2)}(x))^2dx=\sum_{k=1}^{\infty}\{\langle f_n(.),y_k\rangle_{H_1}\}^2. \tag{19}$$

Via integrating by parts twice, we obtain

$$\int_a^c f_n^{(1)}(x)\,y_k^{(1)}(x)\,dx+\gamma\int_c^b f_n^{(2)}(x)\,y_k^{(2)}(x)\,dx$$

$$=\frac{1}{\lambda_k}\int_a^c f_n^{(1)}(x)\left[-\left(py_k^{(1)\prime}\right)'(x)+q(x)\,y_k^{(1)}\right]dx$$

$$+\frac{1}{\lambda_k}\gamma\int_c^b f_n^{(2)}(x)\left[-\left(py_k^{(2)\prime}\right)'(x)+q(x)\,y_k^{(2)}\right]dx$$

$$=\frac{1}{\lambda_k}\int_a^c \left[-\left(pf_n^{(1)\prime}\right)'(x)+q(x)\,f_n^{(1)}\right]y_k^{(1)}dx$$

$$+\frac{1}{\lambda_k}\gamma\int_c^b \left[-\left(pf_n^{(2)\prime}\right)'(x)+q(x)\,f_n^{(2)}\right]y_k^{(2)}dx$$

$$=\frac{1}{\lambda_k}\langle-\left(pf_n'\right)'(.)+q(x)\,f_n(.),y_k\rangle_{H_1}.$$

Thus we have

$$\sum_{|\lambda_k|\geq s}\{\langle f_n(.),y_k\rangle_{H_1}\}^2$$

$$\leq\frac{1}{s^2}\sum_{|\lambda_k|\geq s}\left\{\langle-\left(pf_n'\right)'(.)+q(x)\,f_n(.),y_k\rangle_{H_1}\right\}^2$$

$$\leq\frac{1}{s^2}\sum_{k=1}^{\infty}\left\{\langle-\left(pf_n'\right)'(.)+q(x)\,f_n(.),y_k\rangle_{H_1}\right\}^2$$

$$= \frac{1}{s^2}\int_{-n}^{c}\left[-\left(pf_n^{(1)\prime}\right)'(x)+q(x)\,f_n^{(1)}\right]^2 dx$$

$$+\frac{1}{s^2}\gamma\int_{c}^{n}\left[-\left(pf_n^{(2)\prime}\right)'(x)+q(x)\,f_n^{(2)}\right]^2 dx.$$

By using the formula (19), we obtain

$$\left|\int_{-n}^{c}\left(f_n^{(1)}(x)\right)^2 dx+\gamma\int_{c}^{n}\left(f_n^{(2)}(x)\right)^2 dx-\sum_{-s\leq\lambda_k\leq s}\{\langle f_n(.),y_k\rangle_{H_1}\}^2\right|$$

$$\leq \frac{1}{s^2}\int_{-n}^{c}\left[-\left(pf_n^{(1)\prime}\right)'(x)+q(x)\,f_n^{(1)}\right]^2 dx$$

$$+\frac{1}{s^2}\gamma\int_{c}^{n}\left[-\left(pf_n^{(2)\prime}\right)'(x)+q(x)\,f_n^{(2)}\right]^2 dx.$$

Furthermore, we have

$$\sum_{-s\leq\lambda_k\leq s}\{\langle f_n(.),y_k\rangle_{H_1}\}^2$$

$$=\sum_{-s\leq\lambda_k\leq s}\{\langle f_n(.),c_k\phi_1(.,\lambda_k)+d_k\phi_2(.,\lambda_k)\rangle_{H_1}\}^2$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\mu_{ij,a,b}(\lambda),$$

where

$$F_{in}(\lambda)=\langle f_n(.),\phi_i(.,\lambda)\rangle_{H_1}\ \ (i=1,2).$$

Consequently, we get

$$\left| \begin{array}{l} \int_{-n}^{c}\left(f_n^{(1)}(x)\right)^2 dx+\gamma\int_{c}^{n}\left(f_n^{(2)}(x)\right)^2 dx \\ -\int_{-s}^{s}\sum_{i,j=1}^{2}F_{in}(\lambda)F_{jn}(\lambda)\,d\mu_{ij,a,b}(\lambda) \end{array} \right|$$

$$\le \tfrac{1}{s^2}\int_{-n}^{c}\left[-\left(pf_n^{(1)\prime}\right)'(x)+q(x)f_n^{(1)}\right]^2 dx \tag{20}$$

$$+\tfrac{1}{s^2}\gamma\int_{c}^{n}\left[-\left(pf_n^{(2)\prime}\right)'(x)+q(x)f_n^{(2)}\right]^2 dx.$$

By Lemma 2.1 and Theorems 2.2 and 2.3, we can find sequences $\{a_k\}$ and $\{b_k\}$, where $a_k\to-\infty$ and $b_k\to\infty$, such that the sequence of functions $\mu_{ij,a_k,b_k}(\lambda)$ converges to a monotone function $\mu_{ij}(\lambda)$. Passing to the limit (with respect to $a_k\to-\infty$ and $b_k\to\infty$) in the inequality (20), we get

$$\left|\int_{-n}^{c}(f_n^{(1)}(x))^2dx+\gamma\int_{c}^{n}(f_n^{(2)}(x))^2dx-\int_{-s}^{s}\sum_{i,j=1}^{2}F_{in}(\lambda)F_{jn}(\lambda)\,d\mu_{ij}(\lambda)\right|$$

$$\le \frac{1}{s^2}\int_{-n}^{c}\left[-\left(pf_n^{(1)\prime}\right)'(x)+q(x)f_n^{(1)}\right]^2 dx$$

$$+\frac{1}{s^2}\gamma\int_{c}^{n}\left[-\left(pf_n^{(2)\prime}\right)'(x)+q(x)f_n^{(2)}\right]^2 dx.$$

As $s\to\infty$, we get

$$\int_{-n}^{c}(f_n^{(1)}(x))^2dx+\gamma\int_{c}^{n}(f_n^{(2)}(x))^2dx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{in}(\lambda)F_{jn}(\lambda)\,d\mu_{ij}(\lambda).$$

Now let

$$f(x)=\begin{cases} f^{(1)}(x), & x\in\Omega_1\\ f^{(2)}(x), & x\in\Omega_2\end{cases}$$

be a real-valued function and $f\in H$. Choose functions $\{f_\eta(x)\}$ satisfying the conditions 1-3 and such that

$$\lim_{\eta\to\infty}\int_{-\infty}^{c}(f^{(1)}(x)-f_\eta^{(1)}(x))^2dx+\lim_{\eta\to\infty}\gamma\int_{c}^{\infty}(f^{(2)}(x)-f_\eta^{(2)}(x))^2dx=0.$$

Let

$$F_{i\eta}(\lambda)=\int_{-\infty}^{c}f_\eta^{(1)}(x)\,\phi_i^{(1)}(x,\lambda)\,dx+\gamma\int_{c}^{\infty}f_\eta^{(2)}(x)\,\phi_i^{(2)}(x,\lambda)\,dx\ \ (i=1,2).$$

Then, we have

$$\int_{-\infty}^{c}(f_{\eta}^{(1)}(x))^{2}dx+\gamma\int_{c}^{\infty}(f_{\eta}^{(2)}(x))^{2}dx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{i\eta}(\lambda)F_{j\eta}(\lambda)d\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{c}(f_{\eta_1}^{(1)}(x)-f_{\eta_2}^{(1)}(x))^{2}dx+\gamma\int_{c}^{\infty}(f_{\eta_1}^{(2)}(x)-f_{\eta_2}^{(2)}(x))^{2}dx\rightarrow 0$$

as $\eta_1,\eta_2\rightarrow\infty$, we obtain

$$\int_{-\infty}^{\infty}\sum_{i=1}^{2}[F_{i\eta_1}(\lambda)F_{j\eta_1}(\lambda)-F_{i\eta_2}(\lambda)F_{j\eta_2}(\lambda)]d\mu_{ij}(\lambda)$$

$$=\int_{-\infty}^{c}(f_{\eta_1}^{(1)}(x)-f_{\eta_2}^{(1)}(x))^{2}dx+\gamma\int_{c}^{\infty}(f_{\eta_1}^{(2)}(x)-f_{\eta_2}^{(2)}(x))^{2}dx\rightarrow 0$$

as $\eta_1,\eta_2\rightarrow\infty$. Therefore, there exist limit functions F_i $(i=1,2)$ which satisfy the equality

$$\int_{-\infty}^{c}\left(f^{(1)}(x)\right)^{2}dx+\gamma\int_{c}^{\infty}\left(f^{(2)}(x)\right)^{2}dx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{i}(\lambda)F_{j}(\lambda)d\mu_{ij}(\lambda),$$

by the completeness of the space $L_{\mu}^{2}(\mathbb{R})$.

Now we will show that, for each $i\in\{1,2\}$, the sequence $(K_{\eta i})$ defined by

$$K_{\eta i}(\lambda)=\int_{-\eta}^{c}f^{(1)}(x)\phi_{i}^{(1)}(x,\lambda)dx+\gamma\int_{c}^{\eta}f^{(2)}(x)\phi_{i}^{(2)}(x,\lambda)dx,$$

converges to F_i as $\eta\rightarrow\infty$, in the metric of the space $L_{\mu}^{2}(\mathbb{R})$. Now let g be another function in H. By a similar argument, $G_i(\lambda)$ $(i=1,2)$ can be defined via the function g.

It is obvious that

$$\int_{-\infty}^{c}\left(f^{(1)}(x)-g^{(1)}(x)\right)^{2}dx+\gamma\int_{c}^{\infty}\left(f^{(2)}(x)-g^{(2)}(x)\right)^{2}dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{(F_{i}(\lambda)-G_{i}(\lambda))(F_{j}(\lambda)-G_{j}(\lambda))\right\}d\mu_{ij}(\lambda).$$

Now let

$$g(x)=\begin{cases} f(x), & x\in[-\eta,c)\cup(c,\eta] \\ 0, & \text{otherwise.}\end{cases}$$

Then, we have

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{\left(F_i(\lambda)-K_{\eta i}(\lambda)\right)\left(F_j(\lambda)-K_{\eta j}(\lambda)\right)\right\}d\mu_{ij}(\lambda)$$

$$=\int_{-\infty}^{-\eta}\left(f^{(1)}(x)\right)^2dx+\gamma\int_{\eta}^{\infty}\left(f^{(2)}(x)\right)^2dx\to 0\ (\eta\to\infty),$$

which proves that $(K_{\eta i})$ converges to F_i in $L_{\mu}^{2}(\mathbb{R})$ for each $i\in\{1,2\}$, as $\eta\to\infty$. □

Theorem 2.5. *Suppose that the real-valued functions f and g are in H, and $F_i(\lambda)$ and $G_i(\lambda)$ $(i=1,2)$ are their generalized Fourier transforms, respectively. Then, we have*

$$\int_{-\infty}^{c}f^{(1)}(x)g^{(1)}(x)dx+\gamma\int_{c}^{\infty}f^{(2)}(x)g^{(2)}(x)dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)G_j(\lambda)d\mu_{ij}(\lambda),$$

which is called the generalized Parseval equality.

Proof. It is clear that $F\mp G$ are the transforms of $f\mp g$. Therefore, we have

$$\int_{-\infty}^{c}\left(f^{(1)}(x)+g^{(1)}(x)\right)^2dx+\gamma\int_{c}^{\infty}\left(f^{(2)}(x)+g^{(2)}(x)\right)^2dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(F_i(\lambda)+G_i(\lambda)\right)\left(F_j(\lambda)+G_j(\lambda)\right)d\mu_{ij}(\lambda),$$

$$\int_{-\infty}^{c}\left(f^{(1)}(x)-g^{(1)}(x)\right)^2dx+\gamma\int_{c}^{\infty}\left(f^{(2)}(x)-g^{(2)}(x)\right)^2dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(F_i(\lambda)-G_i(\lambda)\right)\left(F_j(\lambda)-G_j(\lambda)\right)d\mu_{ij}(\lambda).$$

Subtracting one of these equalities from the other one, we get the desired result. □

Theorem 2.6. *Let f be a real-valued function and $f\in H$. Then, the integrals*

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\phi_j(x,\lambda)d\mu_{ij}(\lambda)\ (i,j=1,2) \tag{21}$$

converge to f in H. Consequently, we have the formula

$$f(x)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda),$$

which is called the expansion formula.

Proof. For any positive number s, set

$$f_s(x)=\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda),$$

where

$$f_s(x)=\begin{cases} f_s^{(1)}(x), & x\in\Omega_1\\ f_s^{(2)}(x), & x\in\Omega_2\end{cases}.$$

Now let $g\in H$ be a real-valued function which is equal to zero outside the finite interval $[-\tau,c)\cup(c,\tau]$. Thus we obtain

$$\int_{-\tau}^{c}f_s^{(1)}(x)\,g^{(1)}(x)\,dx+\gamma\int_{c}^{\tau}f_s^{(2)}(x)\,g^{(2)}(x)\,dx=$$

$$\int_{-\tau}^{c}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\phi_j^{(1)}(x,\lambda)\,d\mu_{ij}(\lambda)\right)g^{(1)}(x)\,dx$$

$$+\gamma\int_{c}^{\tau}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,\phi_j^{(2)}(x,\lambda)\,d\mu_{ij}(\lambda)\right)g^{(2)}(x)\,dx$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\left\{\begin{array}{c}\int_{-\tau}^{c}\phi_j^{(1)}(x,\lambda)\,g^{(1)}(x)\,dx\\ +\gamma\int_{c}^{\tau}\phi_j^{(2)}(x,\lambda)\,g^{(2)}(x)\,dx\end{array}\right\}d\mu_{ij}(\lambda)$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda) \tag{22}$$

From Theorem 2.5, we have

$$\int_{-\infty}^{c}f^{(1)}(x)\,g^{(1)}(x)\,dx+\gamma\int_{c}^{\infty}f^{(2)}(x)\,g^{(2)}(x)\,dx$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda). \tag{23}$$

By (22) and (23), we have

$$\int_{-\infty}^{c}(f^{(1)}(x)-f_s^{(1)}(x))g^{(1)}(x)\,dx+\gamma\int_{c}^{\infty}(f^{(2)}(x)-f_s^{(2)}(x))g^{(2)}(x)\,dx$$

$$=\int_{|\lambda|>s}\sum_{i,j=1}^{2}F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda).$$

If we apply this equality to the function

$$g(x)=\begin{cases} f(x)-f_s(x), & x\in[-s,c)\cup(c,s] \\ 0, & \text{otherwise,}\end{cases}$$

then we get

$$\int_{-\infty}^{c}(f^{(1)}(x)-f_s^{(1)}(x))^2dx+\gamma\int_{c}^{\infty}(f^{(2)}(x)-f_s^{(2)}(x))^2dx$$

$$=\int_{|\lambda|>s}\sum_{i,j=1}^{2}F_i(\lambda)\,F_j(\lambda)\,d\mu_{ij}(\lambda).$$

Letting $s\to\infty$ yields the expansion result. □

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